XIII. *On Electrical Motions in a Spherical Conductor.*

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THIS paper treats of the motions of electricity produced in a spherical conductor by any electric or magnetic operations outside it. The investigation was undertaken some time ago in illustration of MAXWELL'S theory of Electricity. This theory is so remarkable, more especially in the part which it assigns to dielectric media in the propagation of electromagnetic effects, that it seemed worth while to attack some problem in which all the details of the electrical processes could be submitted to calculation, although it was evident beforehand, from the researches of HELMHOLTZ[*] and others, that the results (so far as they are peculiar to the theory) would be of far too subtle a character to admit of comparison with experiment. In studying the mathematical character of the problem above stated I was led to a certain system of formulæ which I have since utilised in two communications to the London Mathematical Society,[†] and which seem likely to be of use in a great variety of physical questions.

§ 1 consists mainly of a recital of the fundamental equations and of the conditions to be satisfied at the surface of a conductor. It is assumed, in the first instance, that the magnetic susceptibility of the conductor is zero.

In § 2 is introduced the assumption that all our functions vary as $e^{\lambda t}$, where t is the time, and λ a constant. It is pointed out that this assumption is sufficiently general. The fundamental equations are then put into a mathematically convenient form. Before, however, proceeding to apply these equations as they stand, I examine the effect of assuming that the velocity (v) of propagation of electromagnetic effects in the medium surrounding the conductor is practically infinite. This assumption, which has been made by all writers (including MAXWELL himself) who have applied MAXWELL'S theory to ordinary electromagnetic phenomena, greatly simplifies the calculations without sensibly impairing the practical value of the results. If L stand for a linear dimension of the conductor and ρ for its specific resistance, it will appear in the sequel that when as in all practical cases λ is small compared with v/L, the error introduced by the assumption in question is of the order $\lambda\rho/v^2$. For any ordinary metallic conductor, and for any value of λ which can be appreciated experimentally, this fraction is excessively minute.

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[*] CRELLE, t. 72 (1870).

[†] "On the Oscillations of a Viscous Spheroid," Proc. L. M. S., Nov. 10, 1881; and "On the Vibrations of an Elastic Sphere," May 11, 1882.

In § 3 the solutions of our equations (on the assumption above indicated) are given in the form appropriate to our present problem. These solutions are of two distinct types. Those of the first type, which are much the more important from an experimental point of view, have (I find) been discussed, though by a different method, by Professor C. NIVEN in a paper recently published.[*] As the points to which attention has been directed are for the most part sufficiently distinct in the two investigations, I have allowed the corresponding portions of my paper to stand.

In § 4 I discuss the case of electric currents started anyhow in the sphere and left to themselves. The equation which gives the "moduli" of the natural modes of decay of the first type agrees with the result obtained by Professor NIVEN.

In § 5 is studied the case of induced currents. Since any disturbance in the field (however arbitrary) can be expressed, as regards the time, by a series of simple harmonic terms, it is sufficient to consider the case when the variations in the inducing system follow the simple harmonic law. This case has moreover acquired a special interest since the invention of the telephone.

The two extreme cases, when the period of the variation in the field is very large or very small in comparison with the time of decay of free currents in the sphere, are discussed in some detail.

In § 6 the case of a thin spherical *shell* is briefly examined.

I next proceed to investigate what modifications must be introduced into the methods and the results of the preceding sections when the substance of the sphere is susceptible of magnetisation. This occupies §§ 7, 8, 9, 10.

In the remaining sections of the paper I investigate the solution of our fundamental equations, taking account of the finite value of v. The corrections to our former results are of most interest in the solutions of the second type. Although the preceding theory, based on the assumption $v=\infty$, is sufficient for all purposes of comparison with experiment, there are certain processes of (at all events) theoretical interest of which it fails altogether to give an account, viz., all those cases in which any change in the superficial electrification of the sphere takes place. For the expression of these the solutions of the second type are appropriate. There is no difficulty in working out the requisite formulæ, but in the application to the case of *free* motion a difficulty of interpretation arises which is noticed in the proper place.

1. Let us suppose that we have one or more conductors at rest in an insulating

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[*] Phil. Trans., 1882. The date of the paper is January, 1880.

medium. If F, G, H be the components of electromagnetic momentum, u, v, w those of electric current, at the point (x, y, z), we have on MAXWELL's theory

$$\left.\begin{aligned}\nabla^2\mathrm{F}&=-4\pi u\\ \nabla^2\mathrm{G}&=-4\pi v\\ \nabla^2\mathrm{H}&=-4\pi w\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{d\mathrm{F}}{dx}+\frac{d\mathrm{G}}{dy}+\frac{d\mathrm{H}}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where ∇^2 stands for $d^2/dx^2+d^2/dy^2+d^2/dz^2$. These equations hold good in conductors and insulators alike, provided that (as we shall assume for the present) the magnetic permeability in neither case differs sensibly from unity.

In the conductors we have, if ρ be the specific resistance,

$$\left.\begin{aligned}\rho u&=-\frac{d\phi}{dx}-\frac{d\mathrm{F}}{dt}\\ \rho v&=-\frac{d\phi}{dy}-\frac{d\mathrm{G}}{dt}\\ \rho w&=-\frac{d\phi}{dz}-\frac{d\mathrm{H}}{dt}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The expressions on the right-hand side of (3) are the components of electromotive force, ϕ being a function which, in the case of *steady* motion of electricity, is known by the name of the "electric potential."*

In the dielectric we have, if f, g, h be the components of electric displacement, and $1/v^2$ the specific inductive capacity, measured (like all our quantities) on the electromagnetic system,

$$\left.\begin{aligned}4\pi v^2 f&=-\frac{d\phi}{dx}-\frac{d\mathrm{F}}{dt}\\ 4\pi v^2 g&=-\frac{d\phi}{dy}-\frac{d\mathrm{G}}{dt}\\ 4\pi v^2 h&=-\frac{d\phi}{dz}-\frac{d\mathrm{H}}{dt}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

v is the velocity of propagation of electromagnetic effects in the dielectric medium. If this be air, v also denotes the number of electrostatic units in one electromagnetic unit of electricity.

The conditions to be satisfied at the boundary of a conductor are that F, G, H and their first derivatives must be continuous. This follows at once from the expressions for F, G, H in terms of the electric currents in the field, viz.,

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* In other cases, as will be seen, this name is less appropriate.

$$\left.\begin{aligned} \mathrm{F}&=\iiint\frac{u'}{r}dx'dy'dz' \\ \mathrm{G}&=\iiint\frac{v'}{r}dx'dy'dz' \\ \mathrm{H}&=\iiint\frac{w'}{r}dx'dy'dz' \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

where r denotes the distance from the element $dx'dy'dz'$ to the point (x, y, z) at which the values of F, G, H are required. Hence if a, b, c be the components of magnetic induction, viz.,

$$\left.\begin{aligned} a&=\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz} \\ b&=\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx} \\ c&=\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{F}}{dy} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6),$$

these quantities will be continuous at the surface of a conductor. Conversely we may show that if F, G, H, a, b, c be continuous then the first derivatives of F, G, H will all be continuous. For this it is sufficient to prove that their derivatives in the direction of the normal will be continuous. If l, m, n be the direction-cosines of the normal, we have

$$\begin{aligned} l\frac{d\mathrm{F}}{dx}+m\frac{d\mathrm{F}}{dy}+n\frac{d\mathrm{F}}{dz}&=l\frac{d\mathrm{F}}{dx}+m\frac{d\mathrm{G}}{dx}+n\frac{d\mathrm{H}}{dx}+nb-mc \\ &=\left(m\frac{d\mathrm{G}}{dx}-l\frac{d\mathrm{G}}{dy}\right)-\left(l\frac{d\mathrm{H}}{dz}-n\frac{d\mathrm{H}}{dx}\right)+nb-mc \quad . \; . \; . \quad (7), \end{aligned}$$

by (2), and it is easily seen from geometrical considerations that the continuity of G implies the continuity of $(m.d\mathrm{G}/dx-l.d\mathrm{G}/dy)$, and so on. Hence if F, G, H, a, b, c be continuous, the first member of (7), and the corresponding expressions for the normal derivatives of G and H, are continuous.

From this point the letters u, v, w will be used to denote solely the components of current *in the conductors*. The components of current in the dielectric are $\dot{f}, \dot{g}, \dot{h}$.

The general solenoidal conditions to be satisfied by u, v, w and $\dot{f}, \dot{g}, \dot{h}$, viz.,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8),$$

and

$$\frac{d}{dt}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

require, by (3) and (4),

$$\nabla^2\phi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

in the conductors, and

$$\nabla^2\frac{d\phi}{dt}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

in the dielectric. The superficial solenoidal condition

$$lu+mv+nw=l\frac{df}{dt}+m\frac{dg}{dt}+n\frac{dh}{dt}$$

requires, by (1), the continuity of

$$l\nabla^2\mathrm{F}+m\nabla^2\mathrm{G}+n\nabla^2\mathrm{H},$$

i.e., of

$$\left(n\frac{da}{dy}-m\frac{da}{dz}\right)+\left(l\frac{db}{dz}-m\frac{da}{dz}\right)+\left(m\frac{dc}{dx}-l\frac{dc}{dy}\right);$$

but this is implied in the continuity of a, b, c.

If $d\nu'$, $d\nu''$ be elements of a normal to the surface of a conductor, on the inside and outside respectively, we find from (3) and (4), taking account of the continuity of F, G, H,

$$-\left(\frac{d\phi}{d\nu'}+\frac{d\phi}{d\nu''}\right)=4\pi v^2(lf+mg+nh)-\rho(lu+mv+nw);$$

or, if σ denote the surface density of electricity,

$$-\left(\frac{d\phi}{d\nu'}+\frac{d\phi}{d\nu''}\right)-4\pi v^2\sigma-\rho\frac{d\sigma}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Hence it is only when the currents are *steady* that the relation between ϕ and the free electricity in the field is the same as in electrostatics.*

If T be the kinetic and V the potential energy of the field, we have

$$\mathrm{T}=\tfrac{1}{2}\iiint(\mathrm{F}u+\mathrm{G}v+\mathrm{H}w)dxdydz$$

$$+\tfrac{1}{2}\iiint(\mathrm{F}\dot{f}+\mathrm{G}\dot{g}+\mathrm{H}\dot{h})d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$\mathrm{V}=2\pi v^2\iiint(f^2+g^2+h^2)d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

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* This peculiarity of MAXWELL's theory has been pointed out by C. NIVEN, *loc. cit.*

where, for the moment, the coordinates x, y, z refer to the conductors, and ξ, η, ζ to the dielectric. Let us form the equation of energy for the case where disturbances produced anyhow in the field are left to themselves.

We have

$$\begin{aligned}\frac{d\mathrm{T}}{dt}&=\tfrac{1}{2}\iiint(\dot{\mathrm{F}}u+\dot{\mathrm{F}}u+\text{\&c.})dxdydz\\&+\tfrac{1}{2}\iiint(\dot{\mathrm{F}}\dot{f}+\dot{\mathrm{F}}\dot{f}+\text{\&c.})d\xi d\eta d\zeta\\&=\iiint(\dot{\mathrm{F}}u+\dot{\mathrm{G}}v+\dot{\mathrm{H}}w)dxdydz\\&+\iiint(\dot{\mathrm{F}}f+\dot{\mathrm{G}}g+\dot{\mathrm{H}}h)d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)^*\end{aligned}$$

Substituting the values of $\dot{\mathrm{F}}$, $\dot{\mathrm{G}}$, $\dot{\mathrm{H}}$ from (3) and (4), we find

$$\begin{aligned}\frac{d\mathrm{T}}{dt}&=-\iiint\rho(u^2+v^2+w^2)dxdydz\\&-4\pi v^2\iiint(f\dot{f}+g\dot{g}+h\dot{h})d\xi d\eta d\zeta\\&-\iiint\left(u\frac{d\phi}{dx}+v\frac{d\phi}{dy}+w\frac{d\phi}{dz}\right)dxdydz\\&-\iiint\left(\dot{f}\frac{d\phi}{d\xi}+\dot{g}\frac{d\phi}{d\eta}+\dot{h}\frac{d\phi}{d\zeta}\right)d\xi d\eta d\zeta\end{aligned}$$

The last two integrals disappear in virtue of the solenoidal conditions satisfied by the flow of electricity.[†] Hence

$$\frac{d}{dt}(\mathrm{T}+\mathrm{W})=-\iiint\rho(u^2+v^2+w^2)d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

This expresses that the electrical energy lost is equivalent to the heat generated in the conductors according to JOULE'S law.

2. Now let us suppose that F, G, H, &c., all vary as $e^{\lambda t}$. The electrical motions in the conductors and in the surrounding dielectric may be of two kinds, *free* and *forced.* In the various modes of free motion the corresponding values of λ are real and negative. In the case of forced motion the disturbing force at any point of the field may, by FOURIER'S (double-integral) theorem, be expanded, as regards the time, in a series of periodic terms. The effects of these can then be investigated separately and afterwards superposed. The value of λ corresponding to any one term is $\lambda=2\pi ip$, where p is the frequency, and $i=\sqrt{-1}$.

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[*] This may be deduced from (1) by GREEN'S Theorem. It is a particular case of THOMSON and TAIT, § 313 (*f*).

[†] MAXWELL'S 'Electricity,' § 100*a*.

On the above assumption, (2) become

$$\rho u=-\frac{d\phi}{dx}-\lambda\mathrm{F},\ \&\text{c.},\ \&\text{c.}\quad\ldots\ldots\ldots\quad(17),$$

whence eliminating u, v, w by means of (1) we obtain as the equations to be satisfied in the interior of a conductor

$$\left.\begin{aligned}(\nabla^2+k^2)\mathrm{F}&=-\frac{k^2}{\lambda}\frac{d\phi}{dx}\\(\nabla^2+k^2)\mathrm{G}&=-\frac{k^2}{\lambda}\frac{d\phi}{dy}\\(\nabla^2+k^2)\mathrm{H}&=-\frac{k^2}{\lambda}\frac{d\phi}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(18)$$

and

$$\frac{d\mathrm{F}}{dx}+\frac{d\mathrm{G}}{dy}+\frac{d\mathrm{H}}{dz}=0\quad\ldots\ldots\ldots\quad(19),$$

where

$$k^2=-\frac{4\pi\lambda}{\rho}\quad\ldots\ldots\ldots\quad(20).$$

In the dielectric we have

$$\nabla^2\mathrm{F}=-4\pi\frac{df}{dt}=-4\pi\lambda f,\ \&\text{c.},\ \&\text{c.}$$

$$4\pi v^2 f=-\frac{d\phi}{dx}-\lambda\mathrm{F},\ \&\text{c.},\ \&\text{c.}$$

Eliminating f, g, h we obtain

$$\left.\begin{aligned}(\nabla^2+j^2)\mathrm{F}&=-\frac{j^2}{\lambda}\frac{d\phi}{dx}\\(\nabla^2+j^2)\mathrm{G}&=-\frac{j^2}{\lambda}\frac{d\phi}{dy}\\(\nabla^2+j^2)\mathrm{H}&=-\frac{j^2}{\lambda}\frac{d\phi}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(21),$$

with

$$\frac{d\mathrm{F}}{dx}+\frac{d\mathrm{G}}{dy}+\frac{d\mathrm{H}}{dz}=0\quad\ldots\ldots\ldots\quad(22),$$

where

$$j^2=-\frac{\lambda^2}{v^2}\quad\ldots\ldots\ldots\quad(23).$$

So far our equations are exact. But it appears from various physical analogies (more especially in Acoustics), and it will be verified in the course of this paper, that when the dimensions of the conductors are small compared with j^{-1} the phenomena are sensibly the same as if j were $=0$. Now, in air, $v=3\times10^{10}$ [C.G.S.], whence $j^{-1}=v/i\lambda=3\times10^{10}/i\lambda$. Since λ is proportional to the rapidity of the electrical motions it appears that in all practical cases j^{-1} is very large. We will therefore assume for the present $j=0$, which comes to the same thing as assuming that the velocity of propagation of electromagnetic effects in the dielectric medium is practically infinite. The equations to be satisfied in the neighbourhood of the conductors then are

$$\nabla^2 F=0, \quad \nabla^2 G=0, \quad \nabla^2 H=0 \quad \ldots \ldots \ldots \quad (24)$$

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0 \quad \ldots \ldots \ldots \quad (25).$$

Since $l\nabla^2 F+m\nabla^2 G+n\nabla^2 H$ must be continuous at the surfaces of the conductors it appears at once that on the present assumption we shall have, at those surfaces,

$$lu+mv+nw=0 \quad \ldots \ldots \ldots \quad (26).$$

3. Proceeding now to the special problem of this paper, viz., the case of a solid spherical conductor surrounded by air, let us take the origin of coordinates at the centre of the sphere, and let r denote the distance of any point from the origin. It may be shown, as in the papers on the "Oscillations of a Viscous Spheroid," &c., already referred to, that the solutions of the equations (18), (19), and (24), (25) are of two distinct types, which are quite independent of one another.

First Type. We have

In the conductor:

$$\left.\begin{aligned} F&=\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n \\ G&=\psi_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\chi_n \\ H&=\psi_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\chi_n \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (27),$$

where χ_n is a solid harmonic of positive integral degree n, and the function ψ_n is defined by

$$\psi_n(\zeta)=1-\frac{\zeta^2}{2.2n+3}+\frac{\zeta^4}{2.4.2n+3.2n+5}-, \text{ \&c.}$$

$$=(-)^n 3.5\ldots 2n+1.\left(\frac{d}{\zeta d\zeta}\right)^n\frac{\sin\zeta}{\zeta} \quad \ldots \ldots \ldots \quad (28),$$

from either of which forms we readily deduce

$$\psi_n'(\zeta)=-\frac{\zeta}{2n+3}\psi_{n+1}(\zeta) \quad \ldots \ldots \ldots \quad (29),$$

$$\psi_n(\zeta)+\frac{\zeta}{2n+1}\psi_n'(\zeta)=\psi_{n-1}(\zeta) \quad \ldots \ldots \ldots \quad (30),$$

$$\psi_n(\zeta)-\psi_{n-1}(\zeta)=\frac{\zeta^2}{2n+1.2n+3}\psi_{n+1}(\zeta) \quad \ldots \ldots \quad (31),$$

The equations (27) constitute the complete solution (of the first type) of (18) and (19) subject to the condition of finiteness at the origin. In the absence of this restriction we should have to add to the right-hand sides similar terms in which n is replaced by $-n-1$.*

In the space surrounding the conductor we have

$$\left.\begin{aligned} F&=\left(y\frac{d}{dz}-z\frac{d}{dy}\right)(X_n+X_{-n-1})\\ G&=\left(z\frac{d}{dx}-x\frac{d}{dz}\right)(X_n+X_{-n-1})\\ H&=\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(X_n+X_{-n-1}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

where X_n, X_{-n-1} are solid harmonics of the algebraical degrees indicated by the suffixes.

Since the values (27) of F, G, H make $(\nabla^2+k^2)\,F=0$, &c., &c., it follows by (1) that the components of current inside the sphere are

$$\left.\begin{aligned} u&=\frac{k^2}{4\pi}\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n\\ v&=\frac{k^2}{4\pi}\psi_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\chi_n\\ w&=\frac{k^2}{4\pi}\psi_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\chi_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

The flow of electricity is everywhere perpendicular to the radius vector, and hence $\phi=$const., inside and outside the sphere.

From (27) and (32) we derive :—

Inside the sphere :

$$\left.\begin{aligned} a&=-\left\{(n+1)\psi_{n-1}(kr)\frac{d\chi_n}{dx}-n\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dx}\chi_nr^{-2n-1}\right\}\\ b&=-\left\{(n+1)\psi_{n-1}(kr)\frac{d\chi_n}{dy}-n\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dy}\chi_nr^{-2n-1}\right\}\\ c&=-\left\{(n+1)\psi_{n-1}(kr)\frac{d\chi_n}{dz}-n\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dz}\chi_nr^{-2n-1}\right\} \end{aligned}\right\} \quad . \quad . \quad (34)\dagger$$

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* These terms would be required in treating the case of a hollow spherical shell.

† These formulæ make

$$xa+yb+zc=-n.n+1.\psi_n(kr).\chi_n$$

Outside:

$$\left.\begin{aligned} a &= -(n+1)\frac{dX_n}{dx}+n\frac{dX_{-n-1}}{dx} \\ b &= -(n+1)\frac{dX_n}{dy}+n\frac{dX_{-n-1}}{dy} \\ c &= -(n+1)\frac{dX_n}{dz}+n\frac{dX_{-n-1}}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

In deducing (34) we have made use of (29), (30), and of the known formula

$$x\chi_n=\frac{r^2}{2n+1}\left(\frac{d\chi_n}{dx}-r^{2n+1}\frac{d}{dx}\chi_n r^{-2n-1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

We have now to apply the conditions to be satisfied at the surface of the sphere. If R be the radius, the continuity of F, G, H requires

$$\psi_n(k\mathrm{R}).\chi_n=X_n+X_{-n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

The continuity of a, b, c requires

$$\psi_{n-1}(k\mathrm{R}).\chi_n=X_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),^*$$

with another condition which is, however, implied in (37) and (38). We must bear in mind that in these equations r is supposed put $=\mathrm{R}$ throughout; so that χ_n, X_n, X_{-n-1} are now *surface* harmonics, of order n.

Second type. We have

Inside the sphere:

$$\phi=\phi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\left.\begin{aligned} \mathrm{F} &= -\frac{1}{\lambda}\frac{d\phi_n}{dx}+(n+1)\psi_{n-1}(kr)\frac{d\omega_n}{dx}-n\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dx}\omega_n r^{-2n-1} \\ \mathrm{G} &= -\frac{1}{\lambda}\frac{d\phi_n}{dy}+(n+1)\psi_{n-1}(kr)\frac{d\omega_n}{dy}-n\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dy}\omega_n r^{-2n-1} \\ \mathrm{H} &= -\frac{1}{\lambda}\frac{d\phi_n}{dz}+(n+1)\psi_{n-1}(kr)\frac{d\omega_n}{dz}-n\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dz}\omega_n r^{-2n-1} \end{aligned}\right\} \quad . \quad (40).^\dagger$$

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* The rigorous proof of these, and of similar inferences in the sequel, may be conducted as in § 4 of the paper "On the Vibrations of an Elastic Sphere" already cited.

† These may also be written

$$\mathrm{F}=-\frac{1}{\lambda}\frac{d\phi_n}{dx}+(2n+1)\psi_{n-1}(kr)\frac{d\omega_n}{dx}-n\frac{d}{dx}\left\{\psi_n(kr)\omega_n\right\}, \text{ \&c.}$$

Outside:

$$\phi=\Phi_n+\Phi_{-n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

$$\left.\begin{aligned} \mathrm{F}&=\frac{d\Omega_n}{dx}+\frac{d\Omega_{-n-1}}{dx} \\ \mathrm{G}&=\frac{d\Omega_n}{dy}+\frac{d\Omega_{-n-1}}{dy} \\ \mathrm{H}&=\frac{d\Omega_n}{dz}+\frac{d\Omega_{n-1}}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

Here ϕ_n, Φ_n, Φ_{-n-1}, ω_n, Ω_n, Ω_{-n-1}, are solid harmonics of the algebraical degrees indicated.

These formulæ give

Inside:

$$\left.\begin{aligned} a&=-k^2\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\omega_n \\ b&=-k^2\psi_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\omega_n \\ c&=-k^2\psi_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\omega_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43);$$

Outside:

$$a=0,\ b=0,\ c=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).$$

The sort of reciprocal relation between the formulæ (27) and (34) on the one hand, and (40) and (43) on the other, is very remarkable.

The continuity of F, G, H at the surface of the sphere implies two relations which we shall not require; whilst that of *a*, *b*, *c* involves

$$\psi_n(k\mathrm{R}).\omega_n=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

This result follows also from (26), since

$$\begin{aligned} xu+yv+zw&=-\frac{1}{4\pi}(x\nabla^2\mathrm{F}+y\nabla^2\mathrm{G}+z\nabla^2\mathrm{H}) \\ &=\frac{k^2}{4\pi}.n(n+1)\psi_n(kr).\omega_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46). \end{aligned}$$

4. From this point we must discuss separately the cases of free and forced motion, respectively. First let us take that of *free motion.* We assume that (no matter how) electric currents have been started in the sphere and then left to themselves.

First Type. The equations (24) must now hold not merely in the space immediately surrounding the sphere but right away to infinity. Hence we must have, in (32), $X_n=0$; and thence, by (37),

$$\psi_{n-1}(kR)=0 \qquad (47).$$

The roots of this equation in kR are all real. For the case $n=1$ we have

$$kR/\pi=1,\ 2,\ 3,\ \&c.$$

When $n=2$,

$$kR/\pi=1\cdot4303,\ 2\cdot4590,\ 3\cdot4709,\ \&c. \qquad (48).$$

When $n=3$,

$$kR/\pi=1\cdot8346,\ 2\cdot8950,\ 3\cdot9225,\ \&c. \qquad (49).$$

When the value of k for any particular mode is known, the corresponding value of λ is given by (20). If τ denote the modulus of decay, *i.e.*, the time in which the currents fall to $1/e$ of their original strength, we have

$$\tau=-\lambda^{-1}=\frac{4}{\pi}\cdot\left(\frac{kR}{\pi}\right)^{-2}\cdot\frac{R^2}{\rho} \qquad (50).$$

For any given mode τ is proportional to the square of the radius, and inversely proportional to the specific resistance; a result which may easily be obtained otherwise, by the method of "dimensions."

For a sphere of copper [$\rho=1642$, C.G.S.] the modulus of the slowest mode of decay is

$$\tau=\cdot000775R^2 \text{ second,}$$

the unit of R being the centimetre. For a copper sphere, of the size of the earth [$R=6\cdot37\times10^8$] the corresponding value of τ is very nearly 10,000,000 years.

As regards the nature of the various modes we may observe that the lines of flow of electricity inside the sphere are the intersections of the spheres $r=$const. with the cones $\chi_n/r^n=$const.; in other words, they are the contour lines of the harmonics χ_n on a series of spherical surfaces concentric with the origin. The intensity of the current at any point is proportional to $\psi_n(kr).d\chi_n/de$, when de is an elementary angle at the centre of the sphere in a plane perpendicular to the line of flow passing through the point in question. The direction of the flow changes sign as we cross either the spheres for which $\psi_n(kr)=0$, or the cones for which $d\chi_n/de=0$. The components of the magnetic induction at points outside the sphere are, by (35)

$$\left.\begin{aligned} a&=nR^{2n+1}\psi_n(kR)\frac{d}{dx}\chi_n r^{-2n-1}\\ b&=nR^{2n+1}\psi_n(kR)\frac{d}{dy}\chi_n r^{-2n-1}\\ c&=nR^{2n+1}\psi_n(kR)\frac{d}{dz}\chi_n r^{-2n-1}\end{aligned}\right\} \qquad (51).$$

The simplest and most important case is when $n=1$. This may easily be examined by making $\chi_1=x$. The lines of motion are then all circles having the axis of x as a common axis.

Second Type. It follows from (45) that we must now have

$$\psi_n(k\mathrm{R})=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

In the cases $n=1$, $n=2$, the first few roots of this equation are given by (48), (49), respectively. The values of the modulus of decay corresponding to the various values of k are to be found from (50). In the most persistent mode of the present type the value of τ for a sphere of copper is

$$\tau=\cdot 000379\ \mathrm{R}^2 \text{ second.}$$

As regards the nature of the motion inside the sphere we remark in the first place that since the radial flow is zero at the surface the electric currents form closed circuits. The flow at any point may be resolved into two components, one along, the other at right angles to, the radius vector. The radial component is

$$\frac{k^2}{4\pi}n.\overline{n+1}.\psi_n(kr).\frac{\omega_n}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

The second or transversal component is perpendicular to that cone of the series $\omega_n/r^n=$ const. which passes through the point in question; and its amount is

$$\frac{k^2}{4\pi}\{kr\psi'_n(kr)+(n+1)\psi_n(kr)\}\frac{d\omega_n}{rd\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

where $d\epsilon$ denotes as before an elementary angle at the centre of the sphere in a plane perpendicular to the above-mentioned cone.

When the harmonic ω_n is *zonal*, having the axis of x (say) as axis, the nature of the motion can be very simply expressed by means of a stream-function Ψ. The motion then takes place in a series of planes through the axis of x and is the same in each such plane. If u, $\mathfrak{v}$ be the components of current parallel and perpendicular to x, viz., $\mathfrak{v}=(yv+zw)/\varpi$, where $\varpi=\sqrt{(y^2+z^2)}$, we have

$$u=\frac{1}{\varpi}\frac{d\Psi}{d\varpi},\ \mathfrak{v}=-\frac{1}{\varpi}\frac{d\Psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

where $2\pi\Psi$ is the total flux through the circle whose coordinates are (x, ϖ). Integrating (53) over the segment of the sphere of radius $r=\sqrt{(x^2+\varpi^2)}$ bounded by this circle we find

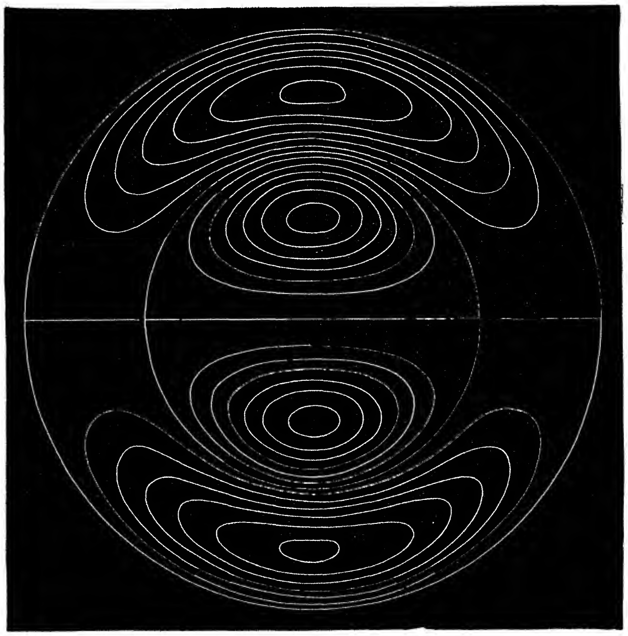
$$\Psi=\frac{k^2}{4\pi}.n.n+1.r\psi_n(kr)\int_0^\theta \omega_n \sin\theta d\theta$$

$$=-\frac{k^2}{4\pi}\psi_n(kr)\varpi\frac{d\omega_n}{d\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (56).$$

Here θ denotes the colatitude (viz., $\varpi=r\sin\theta$), and ω_n is supposed expressed in terms of r, θ. The integration is effected by means of the differential equation of zonal harmonics. The most interesting case is when $n=1$. Writing $\omega_1=r\cos\theta$, we have

$$\Psi=\frac{k^2\varpi^2}{4\pi}\psi_1(kr).$$

The forms of the lines of flow ($\Psi=$ const.) corresponding to a series of equidistant values of Ψ are shown in the figure. The different systems of lines of flow are separated by the spheres for which $\psi_1(kr)=0$. The drawing includes the first two of these. In the most persistent mode the inner sphere must be taken to represent the boundary of the conductor; in the next mode the second sphere must be taken; and so on.



It appears from (44) that the currents in the sphere exercise no magnetic action in the external space. Conversely no motions of the present type can be originated by any electromagnetic operations outside the sphere. It will be shown further on that both these statements require qualification when we take account of the finite value of v.

By combining in the proper way solutions of the two types we can represent the decay of any system of currents arbitrarily given in the sphere. The determination of the harmonics χ_n, ω_n in terms of the initial circumstances, although interesting mathematically, would occupy too much space to be given in full here. It may suffice to remark that if u, v, w be any three functions satisfying the solenoidal condition (8), and if $\xi=dw/dy-dv/dz$, &c., &c., then the values of u, v, w are completely determinate throughout any spherical region having its centre at the origin when we know the values of $xu+yv+zw$ and of $x\xi+y\eta+z\zeta$ throughout that region. This is most readily seen from hydrodynamical considerations. The problem then resolves itself into the identification of the given initial values of these expressions with those which result from our formulæ, viz.,

$$xu+yv+zw=\Sigma\Sigma\frac{k^2}{4\pi}.n.\overline{n+1}.\psi_n(kr)\omega_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56\text{A}),$$

and

$$x\xi+y\eta+z\zeta=-\Sigma\Sigma\frac{k^2}{4\pi}n.\overline{n+1}.\psi_n(kr)\chi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (56\text{B}).$$

The summations here embrace all integral values of n and all admissible values of k. In (56A) these are given by $\psi_n(k\text{R})=0$, and in (56B) by $\psi_{n-1}(k\text{R})=0$. The identification can be effected by known methods.

5. Let us next proceed to consider the currents induced in the sphere by operations outside it; and for simplicity let us suppose that the changes in the field are periodic and follow the simple harmonic law. The value of λ is now prescribed, viz., it $=2\pi ip$, where p is the frequency. Hence, by (20),

$$k^2=-8\pi^2 ip/\rho,$$

and

$$k=(1-i)q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

provided

$$q^2=4\pi^2 p/\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

Since all our formulæ involve only even powers of q there is no loss of generality in taking q always positive.

From (45) we see that $\omega_n=0$, so that we have to deal exclusively with solutions of the *first type*. The complete solution of the problem is then given by the equations (27) and (32) in which the values of χ_n, X_{-n-1} in terms of X_n are to be obtained from the surface-conditions (37) and (38), viz., we have

$$\chi_n=\frac{1}{\psi_{n-1}(k\text{R})}\text{X}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

$$\text{X}_{-n-1}=\left\{\frac{\psi_n(k\text{R})}{\psi_{-n-1}(k\text{R})}-1\right\}\frac{\text{R}^{2n+1}}{r^{2n+1}}\text{X}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

The values of the functions X_n are to be found as follows. It is easily seen that if a_0, b_0, c_0 be the components of the magnetic field due to the inducing system alone, the expression $xa_0+yb_0+zc_0$ must satisfy the equation

$$\nabla^2(xa_0+yb_0+zc_0)=0$$

at all points outside the inducing system, and vanish at the origin. Hence it must admit of expansion in a series of solid harmonics of positive integral degrees, say

$$xa_0+yb_0+zc_0=\Sigma_1^\infty \Theta_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

But it appears from (35) that we must have

$$xa_0+yb_0+zc_0=-\Sigma_1^\infty n.n+1.X_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

Comparing with (61) we find

$$X_n=-\frac{1}{n.n+1}\Theta_n.$$

For instance, let the magnetic field due to the inducing system be sensibly uniform in the neighbourhood of the sphere, say

$$a_0=I,\ b_0=0,\ c_0=0.$$

We find

$$X_1=-\tfrac{1}{2}Ix\,;\ X_2=X_3=\&c.=0.$$

The formulæ (33) for the currents in the sphere then become

$$\left.\begin{aligned} u&=0\\ v&=-D\psi_1(kr).z\\ w&=\ \ D\psi_1(kr).y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

where

$$D=\frac{k^2I}{8\pi\psi_0(kR)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64);$$

and the disturbance (a_1, b_1, c_1) in the magnetic field, due to these currents, is given by

$$\left.\begin{aligned} a_1&=E\frac{d}{dx}\frac{x}{r^3}\\ b_1&=E\frac{d}{dy}\frac{x}{r^3}\\ c_1&=E\frac{d}{dz}\frac{x}{r^3} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65),$$

provided

$$E=-\frac{IR^3}{2}\left\{\frac{\psi_1(kR)}{\psi_0(kR)}-1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

For the full interpretation of our formulæ it would be necessary to disentangle the real and the imaginary parts, and to discard one or the other. The results would be very complicated, even for the simplest harmonic constituent ($n=1$). There are certain cases, however, in which we can use methods of approximation, and so deduce the results of interest without much difficulty.

Thus, in the first place, let us suppose that the changes in the field are comparatively slow; more precisely, let the frequency p be very small compared with ρ/R^2. Since kR is then a small quantity, the expressions for the currents in the sphere are approximately,

$$\left.\begin{aligned} u&=-\frac{2\pi ip}{\rho}\left(y\frac{d}{dz}-z\frac{d}{dy}\right)X_n \\ v&=-\frac{2\pi ip}{\rho}\left(z\frac{d}{dx}-x\frac{d}{dz}\right)X_n \\ w&=-\frac{2\pi ip}{\rho}\left(x\frac{d}{dy}-y\frac{d}{dx}\right)X_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67).$$

This is the result which we should have obtained by neglecting *ab initio* the mutual influence of the currents in the sphere. The disturbance in the field due to these currents is given by

$$\left.\begin{aligned} a_1&=J\frac{d}{dx}X_n r^{-2n-1} \\ b_1&=J\frac{d}{dy}X_n r^{-2n-1} \\ c_1&=J\frac{d}{dz}X_n r^{-2n-1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

where

$$J=\frac{-8\pi^2 inpR^{2n+3}}{2n+1.2n+3.\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69).$$

For spheres of the same size the disturbance is *cæteris paribus* proportional to the specific conductivity.

Next let us examine the other extreme case, where the frequency p is large compared with ρ/R^2, and consequently kR is a large number. When ζ is large, the formula (28) becomes, approximately,

$$\psi_n(\zeta)=(-)^n.3.5\ldots 2n+1.\frac{\sin\left(\zeta+n\dfrac{\pi}{2}\right)}{\zeta^{n+1}}.$$

Writing $\zeta = kr = (1-i)qr$, and keeping only the most important term, we find

$$\psi_n(kr) = (-)^n . 3.5 \ldots 2n+1 . \frac{e^{qr} . e^{i\left(qr + n\frac{\pi}{2}\right)}}{2i(kr)^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70).$$

Hence the factor

$$\frac{k^2}{4\pi}\frac{\psi_n(kr)}{\psi_{n-1}(k\mathrm{R})},$$

which occurs in the expressions for the induced currents, becomes, after several reductions,

$$-\frac{(2n+1)q}{2\sqrt{2\pi\mathrm{R}}} . \left(\frac{\mathrm{R}}{r}\right)^{n+1} . e^{q(r-\mathrm{R}) + i\left\{q(r-\mathrm{R}) + \frac{\pi}{4}\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

It appears from this that the disturbance inside the sphere consists of a series of waves propagated inwards from the surface with rapidly decreasing amplitude. Thus at a depth equal to the wave-length (ν, say), the amplitude is only 1/535 of what it is at the surface. The currents are therefore almost entirely confined to a superficial stratum of thickness comparable with ν. It appears from (58) that ν, $= 2\pi/q$, $= \sqrt{(\rho/p)}$. As a numerical example let $\rho = 1642$ (copper), $p = 4000$; we find

$$\nu = {\cdot}64 \text{ centimetre.}$$

The condition of the applicability of our approximation is that $2\pi\mathrm{R}$ must be large in comparison with ν.*

Since, by (70), $\psi_n(k\mathrm{R})/\psi_{n-1}(k\mathrm{R})$ is of the order $1/k\mathrm{R}$, it appears from (60) and (38) that the disturbance in the field caused by the currents in the sphere is given by

$$\left.\begin{aligned} a_1 &= -n\mathrm{R}^{2n+1}\frac{d}{dx}\,\mathrm{X}_n r^{-2n-1} \\ b_1 &= -n\mathrm{R}^{2n+1}\frac{d}{dy}\,\mathrm{X}_n r^{-2n-1} \\ c_1 &= -n\mathrm{R}^{2n+1}\frac{d}{dz}\,\mathrm{X}_n r^{-2n-1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72).$$

The magnitude of the disturbance depends therefore on the size of the sphere, but is independent of the conductivity, so long as the fundamental condition of our approximation is satisfied. The reason of this is not far to seek. The greater the conductivity the greater will be the intensity of the currents at the surface of the sphere, but the more rapid will be the rate of diminution as we pass inwards; and it is easily seen from (71) that one cause will exactly compensate the other.

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* The above results enable us to estimate what ought to be the thickness of a sheet of a given metal in order that it should act as a screen against a periodic electromagnetic action of given frequency. See the paper by Lord RAYLEIGH, cited below.

In fact, if we write

$$u'=\int^{\mathrm{R}} udr,\quad v'=\int^{\mathrm{R}} vdr,\quad w'=\int^{\mathrm{R}} wdr,$$

where the lower limit is taken at such a depth that the currents there are insensible, we readily find that the currents are approximately equivalent to an infinitely thin spherical *current sheet* of radius R, the components of the current at any point of the sheet being given by

$$\left.\begin{aligned} u'&=-\frac{2n+1}{4\pi\mathrm{R}}\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\mathrm{X}_n \\ v'&=-\frac{2n+1}{4\pi\mathrm{R}}\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\mathrm{X}_n \\ w'&=-\frac{2n+1}{4\pi\mathrm{R}}\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\mathrm{X}_n \end{aligned}\right\}\quad \ldots\ldots\ldots \quad (73).^*$$

6. The foregoing methods can be readily adapted to the case of a shell bounded by two concentric spherical surfaces. The most interesting case is when the shell is infinitely thin. The free motions of the second type then decay with infinite rapidity, and there are no forced motions of this type. Hence we have practically to deal only with solutions of the first type. The theory of these has been given by Professor NIVEN, but for the sake of completeness it is here discussed from the point of view of the present paper.

Let u', v', w' be the components of the total current at any point of the shell, and let $\rho'=\rho/\delta$, where δ is the thickness of the shell. Then if all our functions vary as $e^{\lambda t}$ we shall have

$$\rho'u'=-\lambda\mathrm{F},\quad \rho'v'=-\lambda\mathrm{G},\quad \rho'w'=-\lambda\mathrm{H}\quad \ldots\ldots \quad (74).$$

In the hollow space inside the shell

$$\left.\begin{aligned} \mathrm{F}&=\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n \\ \mathrm{G}&=\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\chi_n \\ \mathrm{H}&=\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\chi_n \end{aligned}\right\}\quad \ldots\ldots\ldots \quad (75),^\dagger$$

whilst (32) hold for the external space. The functions F, G, H must vary continually as we cross the shell, so that

$$\chi_n=\mathrm{X}_n+\mathrm{X}_{-n-1}\quad \ldots\ldots\ldots\ldots \quad (76),$$

at the surface.

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[*] The conclusions of this section have an obvious bearing on the results obtained by Professor D. E. HUGHES in his experiments with the Induction Balance (Proc. Roy. Soc., May 15, 1879).

[†] It is here assumed that the inducing system, if any, is situate in the space external to the shell.

The first derivatives of F, G, H are, however, discontinuous, viz., if $d\nu'$, $d\nu''$ be elements of the normal drawn inwards and outwards respectively, we must have

$$\left.\begin{aligned}\frac{dF}{d\nu'}+\frac{dF}{d\nu''}&=-4\pi u'\\ \frac{dG}{d\nu'}+\frac{dG}{d\nu''}&=-4\pi v'\\ \frac{dH}{d\nu'}+\frac{dH}{d\nu''}&=-4\pi w'\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (77),$$

which equations now replace (1). Hence, and from (74) we deduce

$$-n\chi_n+nX_n-(n+1)X_{-n-1}=\frac{4\pi R\lambda}{\rho'}\chi_n \quad \ldots\ldots \quad (78),$$

when $r=R$, the radius of the shell.

In free motion $X_n=0$, and thence

$$\tau=-\lambda^{-1}=\frac{4\pi R}{(2n+1)\rho'} \quad \ldots\ldots\ldots \quad (79).$$

In the case of currents induced by a system external to the shell, we find

$$\chi_n=\frac{1}{1+\lambda\tau}X_n \quad \ldots\ldots\ldots \quad (80),$$

and

$$X_{-n-1}=-\frac{\lambda\tau}{1+\lambda\tau}X_n \quad \ldots\ldots\ldots \quad (81),$$

when τ has the value (79). The value of X_n can be found as before when the nature of the inducing system is known. Writing $\lambda=2\pi ip$ we see from (80) that if the period of the disturbance be small compared with τ the shell will almost completely shelter the enclosed region from the electromagnetic action of the external system.*

The case where the inducing system is inside the shell may be treated in a similar manner. We have to introduce a function χ_{-n-1} for the internal space, whilst X_n is zero.

7. When the magnetic permeability μ of the substance of the conductor differs sensibly from unity, the processes of the foregoing articles require some modification. The equations (1) must then be replaced by

$$\left.\begin{aligned}\nabla^2F&=-4\pi\mu u\\ \nabla^2G&=-4\pi\mu v\\ \nabla^2H&=-4\pi\mu w\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (82),$$

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* See Lord RAYLEIGH, Phil. Mag., May, 1882, p. 344.

whilst (2) and (3) are unaltered. Hence the fundamental equations (18), (19) of our method retain the same form, provided

$$k^2=-\frac{4\pi\mu\lambda}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83).$$

The distribution of the induced magnetization in the conductor will be solenoidal. Hence if A, B, C be the components of this distribution, the corresponding parts of F, G, H will be

$$\frac{d\text{N}}{dy}-\frac{d\text{M}}{dz}, \ \frac{d\text{L}}{dz}-\frac{d\text{N}}{dx}, \ \frac{d\text{M}}{dx}-\frac{d\text{L}}{dy},$$

respectively, where

$$\text{L}=\iiint\frac{\text{A}}{r}\,dxdydz, \ \text{M}=\iiint\frac{\text{B}}{r}\,dxdydz, \ \text{N}=\iiint\frac{\text{C}}{r}\,dxdydz.$$

The integrations are supposed to extend throughout the magnetized substance, and r denotes the distance between the element $dxdydz$ and the point for which the values of L, M, N are required. Hence F, G, H are continuous at the surface, but their first derivatives, and consequently a, b, c, will be discontinuous. Let us distinguish the values of a, b, c just inside and just outside the conductor by the accents $'$ and $''$, respectively. Then the parts of a', b', c' due to the induced magnetisation are

$$-\mu\frac{d\text{V}}{dx}, \ -\mu\frac{d\text{V}}{dy}, \ -\mu\frac{d\text{V}}{dz},$$

and those of a'', b'', c'' are

$$-\frac{d\text{V}}{dx}, \ -\frac{d\text{V}}{dy}, \ -\frac{d\text{V}}{dz},$$

where V is the potential of free magnetism, viz.:

$$\text{V}=\iint(l\text{A}+m\text{B}+n\text{C})\frac{d\text{S}}{r},$$

dS denoting an element of the surface of the body, and l, m, n the direction-cosines of the outwardly directed normal to dS, and the integration being taken over the surface of the conductor. Hence

$$a''-\frac{a'}{\mu}=4\pi l(l\text{A}+m\text{B}+n\text{C}), \text{ \&c.};$$

or, since $4\pi\mu\text{A}=(\mu-1)a'$, &c.,

$$\left.\begin{aligned} a'+(\mu-1)\,l\,(la'+mb'+nc')&=\mu a'' \\ b'+(\mu-1)m(la'+mb'+nc')&=\mu b'' \\ c'+(\mu-1)\,n\,(la'+mb'+nc')&=\mu c'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84).$$

We notice that these conditions give

$$la'+mb'+nc'=la''+mb''+nc'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85),$$

as ought to be the case. In fact (85) is implied in the continuity of F, G, H.

8. Proceeding now to the case of a spherical conductor, let the origin be taken at the centre, and let r be the radius vector of any point. Let us begin with the solutions of the *First Type*, the formulæ for which are given by (27), (32), (34), (35). The continuity of F, G, H gives as before

$$\psi_n(k\mathrm{R}).\chi_n=\mathrm{X}_n+\mathrm{X}_{-n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

at the surface ($r=\mathrm{R}$). In applying (84) we remark that, at the surface,

$$la'+mb'+nc'=-\frac{n.n+1}{\mathrm{R}}\psi_n(k\mathrm{R}).\chi_n,$$

and hence that

$$l(la'+mb'+nc')=-\frac{n.n+1}{\mathrm{R}^2}\psi_n(k\mathrm{R}).x\chi_n,$$

$$=-\frac{n.n+1}{2n+1}\psi_n(k\mathrm{R})\left(\frac{d\chi_n}{dx}-\mathrm{R}^{2n+1}\frac{d}{dx}\chi_n r^{-2n-1}\right),$$

by (36). Hence (84) give

$$\left\{\psi_{n-1}(k\mathrm{R})+\frac{n}{2n+1}(\mu-1)\psi_n(k\mathrm{R})\right\}\chi_n=\mu\mathrm{X}_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (87),$$

with another condition which may however be shown to be included in (86) and (87).

The formulæ for the solutions of the *Second Type* are given in (40), (42), (43), (44). The surface conditions (84) yield

$$\psi_n(k\mathrm{R}).\omega_n=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88).$$

9. In the case of *free* currents of the first type we have $\mathrm{X}_n=0$, and the equation to determine k is

$$\psi_{n-1}(k\mathrm{R})+\frac{n}{2n+1}(\mu-1)\psi_n(k\mathrm{R})=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89).$$

When, as in iron, μ is a very large number, we have, as a first approximation,

$$\psi_n(k\mathrm{R})=0.$$

If $k\mathrm{R}=\vartheta$ be a solution of this, a second approximation is

$$k\mathrm{R}=\left\{1-\frac{1}{n(\mu-1)}\right\}\vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90).$$

When the values of $k\mathrm{R}$ have been found, the corresponding values of the modulus of decay are given by

$$\tau=\frac{4}{\pi}\left(\frac{k\mathrm{R}}{\pi}\right)^{-2}\frac{\mu\mathrm{R}^2}{\rho}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91).$$

In iron we have $\mu=403$ (**Thalén**), $\rho=9827$ C. G. S. The lowest root of (89), in the case $n=1$, is then $k\mathrm{R}=1\cdot4268\pi$, and the corresponding value of τ is

$$\tau=\cdot0256\mathrm{R}^2.$$

The duration of the free currents is very much greater than in a non-magnetizable sphere of the same size and of equal conductivity. For an iron ball one foot in diameter the above value of τ is six seconds. For an iron globe of the size of the earth it would be 330,000,000 years.

The magnetic susceptibility of the substance has the effect of modifying the character, as well as the duration, of the natural modes of decay. Inside the sphere we have

$$la+mb+nc=-\frac{n.\overline{n+1}}{\mathrm{R}}.\psi_n(kr)\chi_n,$$

Since, by (89), this is almost zero at the surface, the lines of magnetic induction inside the sphere are for the most part closed curves. Their forms, in the first two modes of the class $n=1$, are given by the figure of § 4. The surface of the conductor is not, however, in these two respective modes, now represented by the two spherical surfaces there shown, but rather by two concentric spherical surfaces of radii smaller (for the case of iron) by about the four hundredth part.

For the free currents of the second type we have, by (88),

$$\psi_n(k\mathrm{R})=0\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92).$$

The natural modes of decay are exactly the same as when $\mu=1$, but the persistency is in each case greater in the ratio of $\mu:1$; viz., the values of τ corresponding to the various roots of (92) are given by (91).

10. In the case of *induced* currents caused by a periodic variation in the magnetic field the value of X_n is to be found in the same manner as in § 5; and χ_n, X_{-n-1} are then determined by (86) and (87). If p be the frequency,

$$k=(1-i)q$$

where now

$$q^2=4\pi^2\mu p/\rho\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93).$$

Let us examine first the case where $k\mathrm{R}$ is small. We then have, at the surface,

$$\chi_n=\frac{\overline{2n+1}.\mu}{n\mu+n+1}\mathrm{X}_n\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94),$$

$$X_{-n-1}=\frac{(n+1)(\mu-1)}{n\mu+n+1}X_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

approximately. The currents in the sphere are then given by

$$u=-\frac{2\pi ip}{\rho}\frac{2n+1.\mu^2}{n\mu+n+1}\left(y\frac{d}{dz}-z\frac{d}{dy}\right)X_n,\ v=\&c.,\ w=\&c. \quad . \quad . \quad . \quad . \quad (96).$$

The principal part of the disturbance in the field, due to the presence of the sphere, is given by

$$a_1=\frac{n(n+1)(\mu-1)}{n\mu+n+1}\frac{d}{dx}\frac{R^{2n+1}}{r^{2n+1}}X_n,\ b_1=\&c.,\ c_1=\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

These terms express the effect of the induced magnetization of the sphere. The effect of the induced currents is (under the circumstances supposed) small in comparison.

Next let us take the case of kR large. It is to be noticed that owing to the occurrence of the factor μ in (93) this condition is satisfied by very much smaller values of the frequency than the case of a non-magnetizable substance. We then have

$$\frac{k^2}{4\pi}\psi_n(kr)\chi_n=\frac{k^2\mu}{4\pi}\cdot\frac{2n+1.\psi_n(kr)}{2n+1.\psi_{n-1}(kR)+(\mu-1)n\psi_n(kR)}X_n.$$

The factor of X_n is by (70)

$$=\frac{2n+1}{4\pi}\cdot\frac{k^2\mu}{n(\mu-1)+ikR}\cdot\left(\frac{R}{r}\right)^{n+1}\cdot e^{q(r-R)+iq(r-R)},$$

approximately. If we assume

$$n(\mu-1)+qR=D\cos\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

$$qR=D\sin\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99),$$

this may be written

$$-\frac{2n+1}{2\pi}\cdot\frac{q^2\mu}{D}\cdot\left(\frac{R}{r}\right)^{2n+1}\cdot e^{q(r-R)+i\left\{q(r-R)+\frac{\pi}{2}-\epsilon\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (100).$$

From this result we draw conclusions similar to those of § 5. The depth ν within which the maximum intensity of current falls to $1/e$ of its surface values is

$$\nu=2\pi/q=\sqrt{\frac{\rho}{\mu p}}.$$

In the case of iron we have, using the same data as before, $\nu=\cdot078$ centim. for a frequency of 4000, or $\nu=\cdot78$ for a frequency of 40. The value of ν is thus, for the same frequency, very much smaller than in copper. But the integral currents induced, under the same circumstances, are much more intense in an iron sphere than in a copper sphere of the same size. Integrating (100) with respect to r through the thickness of the stratum in which the currents are sensible, we find for the components of flow at any point of the equivalent *current sheet*

$$\left.\begin{aligned} u' &= \mathrm{K}\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\mathrm{X}_n \\ v' &= \mathrm{K}\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\mathrm{X}_n \\ w' &= \mathrm{K}\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\mathrm{X}_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101),$$

where

$$\mathrm{K}=-\frac{2n+1}{2\sqrt{2\pi}\mathrm{R}}\cdot\frac{q\mathrm{R}}{\mathrm{D}}\cdot\mu . e^{i\left(\frac{\pi}{4}-\epsilon\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102).$$

The disturbance in the field, due to the presence of the sphere, is given by

$$a_1=n\frac{d\mathrm{X}_{-n-1}}{dx},\ b_1=\&\mathrm{c}.,\ c_1=\&\mathrm{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103),$$

where

$$\mathrm{X}_{-n-1}=\left[\frac{2n+1.\mu}{\mathrm{D}}e^{-i\epsilon}-1\right]\frac{\mathrm{R}^{2n+1}}{r^{2n+1}}\mathrm{X}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104).$$

The order of magnitude of the first term within [ ] depends on the relative magnitudes of qR and μ. So long as qR, though itself large, is moderately small in comparison with $n\mu$ the effect is mainly due to the induced magnetization of the sphere, and is much the same as if the substance were destitute of electrical conductivity, although the distribution of the magnetization within the sphere is very different. On the other hand when qR is large compared with $n\mu$ the first term in [ ] is less important, and the results approximate more to the form which they would assume in the case of infinite conductivity. The following table gives the values of D and ϵ for iron, in the case $n=1$, corresponding to various values of qR.

qR.	10.	50.	100.	1000.
D ϵ	412 1°.23′	455 6°.19′	512 11°.16′	1722 35°.30′

The relation between q and the frequency p is for iron

$$q=1{\cdot}27\sqrt{p}.$$

11. In the whole of the preceding investigations it has been assumed that the quantity j of § 2 may without sensible error be put $=0$. I proceed to sketch the method to be pursued when we do not make this assumption, confining myself for simplicity to the case of $\mu=1$ everywhere. The fundamental equations to be satisfied are:—for the spherical conductor (18) and (19); for the surrounding dielectric (21) and (22).

In the solution of the *First Type* the values of F, G, H and of a, b, c inside the sphere are then given by (27) and (34), respectively. Outside the sphere we shall now have

$$\mathrm{F}=\psi_n(jr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\mathrm{X}_n+\psi_{-n-1}(jr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\mathrm{X}_{-n-1} \quad . \quad . \quad . \quad (105),$$

where X_n, X_{-n-1} are solid harmonics of the degrees indicated. The values of G and H may be written down from symmetry. We thence find

$$a=-\left\{(n+1)\psi_{n-1}(jr)\frac{d\mathrm{X}_n}{dx}-n\frac{j^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(jr)\frac{d}{dx}\mathrm{X}_nr^{-2n-1}\right\}+\text{terms in }\mathrm{X}_{-n-1}\,.\,(106),$$

with symmetrical formulæ for b, c. The "terms in X_{-n-1}" are to be derived from the preceding line by writing $-n-1$ for n throughout.

The continuity of F, G, H at the surface of the sphere requires

$$\psi_n(k\mathrm{R})\chi_n=\psi_n(j\mathrm{R})\mathrm{X}_n+\psi_{-n-1}(j\mathrm{R})\mathrm{X}_{-n-1} \quad . \quad . \quad . \quad . \quad . \quad (107),$$

when $r=\mathrm{R}$. The continuity of a, b, c requires in addition

$$\psi_{n-1}(k\mathrm{R})\chi_n=\psi_{n-1}(j\mathrm{R})\mathrm{X}_n+\frac{j^2\mathrm{R}^2}{2n-1.2n+1}\psi_{-n}(j\mathrm{R})\mathrm{X}_{-n-1} \quad . \quad . \quad . \quad (108).$$

In the solutions of the *Second Type* the forms of F, G, H, a, b, c inside the sphere are given as before by (40) and (43); whilst in the dielectric we shall now have

$$\phi=\Phi_n+\Phi_{-n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109),$$

and

$$\mathrm{F}=-\frac{1}{\lambda}\frac{d\Phi_n}{dx}-\frac{1}{\lambda}\frac{d\Phi_{-n-1}}{dx}+n+1.\psi_{n-1}(jr)\frac{d\Omega_n}{dx}-n\frac{j^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(jr)\frac{d}{dx}\Omega_nr^{-2n-1}$$
$$+\text{ terms in }\Omega_{-n-1} \quad . \quad (110),$$

with symmetrical formulæ for G, H. The symbols Φ_n, Φ_{-n-1}, Ω_n, Ω_{-n-1} stand for solid harmonics of the algebraical degrees indicated by the suffixes. The foregoing expressions make

$$a=-j^2\psi_n(jr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\Omega_n+\text{ terms in }\Omega_{-n-1}\quad . \quad . \quad . \quad . \quad . \quad (111),$$

with symmetrical formulæ for b and c.

The continuity of ϕ at the surface requires

$$\phi_n=\Phi_n+\Phi_{-n-1}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112),$$

when $r=\text{R}$. The continuity of F, G, H requires

$$\begin{aligned}&-\frac{\phi_n}{\lambda}+(n+1)\psi_{n-1}(k\text{R})\omega_n\\&=-\frac{\Phi_n}{\lambda}+(n+1)\psi_{n-1}(j\text{R})\Omega_n+(n+1)\frac{j^2\text{R}^2}{2n-1.2n+1}\psi_{-n}(j\text{R})\Omega_{-n-1}\quad . \quad . \quad (113),\end{aligned}$$

and

$$-n\frac{k^2\text{R}^2}{2n+1.2n+3}\psi_{n+1}(k\text{R})\omega_n=-\frac{\Phi_{-n-1}}{\lambda}-n\frac{j^2\text{R}^2}{2n+1.2n+3}\psi_{n+1}(j\text{R})\Omega_n-n\psi_{-n-2}(j\text{R})\Omega_{-n-1}\,.\,(114).$$

Adding (113) and (114), and taking account of (112), we find

$$\begin{aligned}&\{k\text{R}\psi'_n(k\text{R})+(n+1)\psi_n(k\text{R})\}\omega_n\\&=\{j\text{R}\psi'_n(j\text{R})+(n+1)\psi_n(j\text{R})\}\Omega_n+\{j\text{R}\psi'_{-n-1}(j\text{R})-n\psi_{-n-1}(j\text{R})\}\Omega_{-n-1}\quad (115),^*\end{aligned}$$

where some reductions have been effected by means of (29) and (30).

The continuity of a, b, c requires

$$k^2\text{R}^2\psi_n(k\text{R})\omega_n=j^2\text{R}^2\psi_n(j\text{R})\Omega_n+j^2\text{R}^2\psi_{-n-1}(j\text{R})\Omega_{-n-1}\,.\quad . \quad . \quad . \quad (116).$$

12. Let us now apply the foregoing results to the case of *free* motion. A certain relation must then hold between the surface values of X_n and X_{-n-1}, and also between those of Ω_n and Ω_{-n-1}, viz.: a relation expressing that the disturbance at infinity in the dielectric is finite. It may be shown that F, G, H are determinate when the values of $x\text{F}+y\text{G}+z\text{H}$ and of $xa+yb+zc$ are known at every point of space. Now in the first type we have $x\text{F}+y\text{G}+z\text{H}=0$, and

$$xa+yb+zc=-n.n+1.\{\psi_n(jr)\text{X}_n+\psi_{-n-1}(jr)\text{X}_{-n-1}\}\quad . \quad . \quad . \quad (117).$$

For large values of r we have

$$\psi_n(jr)=(-)^n.3.5\ldots2n+1.\frac{\sin\left(jr+n\frac{\pi}{2}\right)}{(jr)^{n+1}}\quad . \quad . \quad . \quad . \quad . \quad . \quad (118).$$

---

* Equations (109) and (115) express that the tangential components of current just outside and just inside the sphere are in the ratio of j^2 to k^2. This may also be easily deduced from the fundamental equations (18) and (21).

The last line of (28) is not a convenient expression when n is negative. But we readily deduce from (30)

$$\zeta^{-2n-1}\psi_{-n-1}(\zeta)=-\frac{1}{2n-1}\cdot\frac{d}{\zeta d\zeta}[\zeta^{-(2n-1)}\psi_{-n}(\zeta)] \quad . \quad . \quad . \quad . \quad . \quad (119);$$

and by successive applications of this formula of reduction we find

$$\psi_{-n-1}(\zeta)=\frac{(-)^n\zeta^{2n+1}}{1.3\ldots 2n-1}\cdot\left(\frac{d}{\zeta d\zeta}\right)^n\frac{\cos\zeta}{\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120);$$

since, by (28), $\psi_{-1}(\zeta)=\cos\zeta$. This result, like (28), has been given in somewhat different forms by various writers.* When r is large it gives

$$\psi_{-n-1}(jr)=\frac{(-)^n(jr)^n}{1.3\ldots 2n-1}\cos\left(jr+n\frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (121).$$

In free motion λ is real and negative. We may therefore write $j=-i\lambda/v=i\gamma$ where γ is real, and may be taken positive. Substituting in (118) and (121), and expressing that the terms in $e^{\gamma r}$ must disappear from (117), we are led to the following relation between the surface values of X_n and X_{-n-1}

$$3.5\ldots 2n+1.X_n-i\frac{(jR)^{2n+1}}{1.3\ldots 2n-1}X_{-n-1}=0 \quad . \quad . \quad . \quad . \quad . \quad (122).$$

Similarly in the free motions of the second type we must have

$$3.5\ldots 2n+1.\Omega_n-i\frac{(jR)^{2n+1}}{1.3\ldots 2n-1}\Omega_{-n-1}=0 \quad . \quad . \quad . \quad . \quad . \quad (123).$$

The equation to determine the various values of λ is to be obtained, in the first type by elimination of χ_n, X_n, X_{-n-1} between (107), 108), and (122), and in the second type by elimination of ω_n, Ω_n, Ω_{-n-1} between (115), (116), and (123).

In all practical cases jR is exceedingly small. If we neglect all powers of jR above the second we have

$$X_n=0, \quad \Omega_n=0.$$

In the first type we then obtain

$$\psi_{n-1}(kR)=\frac{j^2R^2}{2n-1.2n+1}\psi_n(kR) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124),$$

approximately. For a first approximation $kR=\vartheta$, where ϑ is a root of $\psi_{n-1}(\vartheta)=0$; and for a second

$$kR=\vartheta\left(1-\frac{1}{2n-1}\frac{j^2}{k^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125).$$

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* See C. Niven, Phil. Trans., 1880, p. 126. Also Heine, 'Kugelfunctionen,' t. i., § 60.

Now $j^2/k^2=\lambda\rho/4\pi v^2$, and $\lambda=-k^2\rho/4\pi=-\rho\vartheta^2/4\pi R^2$, approximately. Hence (125) becomes

$$kR=\vartheta\left\{1+\frac{\rho^2\vartheta^2}{16(2n-1)\pi^2v^2R^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (126)$$

In the second type we have

$$\frac{k^2R^2\psi_n(kR)}{kR\psi'_n(kR)+(n+1)\psi_n(kR)}=-\frac{j^2R^2}{n},$$

or

$$\psi_n(kR)=-\frac{j^2}{nk^2}kR\psi'_n(kR) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (127),$$

to the same degree of accuracy. For a first approximation $kR=\vartheta$, a root of $\psi_n(\vartheta)=0$, and for a second

$$\begin{aligned} kR&=\vartheta\left(1-\frac{j^2}{nk^2}\right) \\ &=\vartheta\left(1+\frac{\rho^2\vartheta^2}{16n\pi^2v^2R^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (128). \end{aligned}$$

By combining together in the proper way solutions of this type we should be able to represent analytically the decay of any given non-uniform electrification of the surface of the sphere. The formula (128) would indicate that in any particular mode the lines of flow of electricity in the sphere are for the most part closed curves, all those which abut on the surface being confined to a stratum of thickness $\rho^2\vartheta^2/16n\pi^2v^2R$. For $n=1$, and $\vartheta/\pi=1\cdot4303$, this $=1\cdot42\times10^{-22}\times\rho^2R^{-1}$. In the case of any ordinary metallic conductor this would be much smaller than the dimensions of a molecule.* A result of this character cannot of course be interpreted literally. All that we can safely assert is that the currents by which the redistribution of the superficial electrification is effected are confined to a very thin film, and are probably subject to laws not yet investigated.

In the case of a globe of water [$\rho=7\cdot18\times10^{10}$ at 22° C.] the result is more intelligible; viz., the thickness of the stratum in question is then$=\cdot73R^{-1}$.

13. The case of periodic induced currents [$\lambda=2\pi ip$ where p is prescribed] may be treated as follows. Let P, Q, R denote the components of electromotive force, viz.:

$$P=-\frac{d\phi}{dx}-\lambda F,\ Q=\&c.,\ R=\&c.$$

It is easily seen that if the suffix $_0$ be used to distinguish the parts of a, b, c, P, Q, R due to the inducing system, the functions $xa_0+yb_0+zc_0$ and $xP_0+yQ_0+zR_0$ must admit of expansion (in the neighbourhood of the origin) in the forms

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* There is nothing peculiar to MAXWELL's theory in the order of magnitude of this result.

$$xa_0+yb_0+zc_0=\Sigma\psi_n(jr)\Theta_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129),$$

$$x\mathrm{P}_0+y\mathrm{Q}_0+z\mathrm{R}_0=\Sigma\psi_n(jr)\mathrm{Z}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130),$$

where Θ_n, Z_n are solid harmonics of positive degree n. Now $xa+yb+zc$ and $x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}$ must differ from the above by terms representing a disturbance propagated wholly outwards. But

$$xa+yb+zc=-n.n+1.\{\psi_n(jr)\mathrm{X}_n+\psi_{-n-1}(jr)\mathrm{X}_{-n-1}\},$$
$$x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}=-\lambda.n.n+1.\{\psi_n(jr)\Omega_n+\psi_{-n-1}(jr)\Omega_{-n-1}\}.$$

The condition that

$$\psi_n(jr)\left(\mathrm{X}_n+\frac{1}{n.n+1}\Theta_n\right)+\psi_{-n-1}(jr)\mathrm{X}_{-n-1}$$

should represent a disturbance travelling outwards may be shown to be

$$3.5\ldots 2n+1\left(\mathrm{X}_n+\frac{1}{n.n+1}\Theta_n\right)+i\frac{(j\mathrm{R})^{2n+1}}{1.3\ldots 2n-1}\mathrm{X}_{-n-1}=0 \quad . \quad . \quad . \quad (131),$$

where in the harmonics X_n, &c., r is supposed put $=\mathrm{R}$. Similarly we have, on the same understanding,

$$3.5\ldots 2n+1\left(\Omega_n+\frac{1}{n.n+1}\frac{\mathrm{Z}_n}{\lambda}\right)+i\frac{(j\mathrm{R})^{2n+1}}{1.3\ldots 2n-1}\Omega_{-n-1}=0 \quad . \quad . \quad . \quad (132).$$

The equations (107), (108), and (131) determine χ_n, X_n, X_{-n-1} in terms of Θ_n; whilst (115), (116), and (132) determine ω_n, Ω_n, Ω_{-n-1} in terms of Z_n. Thus the complete solution of our problem is effected.

Introducing the consideration that $j\mathrm{R}$ is small, we find, in the solutions of the second type,

$$\Omega_n=-\frac{1}{n.n+1}\frac{\mathrm{Z}_n}{\lambda},$$

approximately, and thence

$$k^2\psi_n(kr)\omega_n=-\frac{2n+1}{n^2.n+1}\frac{j^2}{\lambda}\mathrm{Z}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133),$$

by (115) and (116). If σ_n denote the n^{th} harmonic constituent of the surface distribution of electricity, we deduce

$$\sigma_n=\frac{1}{\lambda}\frac{d\sigma_n}{dt}=\frac{1}{4\pi v^2\mathrm{R}}\cdot\frac{2n+1}{n}\cdot\mathrm{Z}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134).$$

For the first harmonic constituent we have the simple formula

$$\sigma_1 = \frac{3(xf' + yg' + zh')}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135),$$

if f', g', h' denote the components of the electric displacement which would obtain at the origin if the spherical conductor were removed.

The equation (134) expresses that so long as $p\rho$ is small compared with v^2 the surface-density of electricity at any point will have at each instant the statical value corresponding to the distribution of electromotive force at that instant due to the external system. The arrangement of the currents in the sphere by which the changes in the superficial distribution are effected will however depend materially on the relation between the period of the changes in the field and the time of decay of free currents in the sphere. The discussion of this point can be conducted as in the case of the solutions of the first type, treated in § 5, and the results are analogous to those there found. When the spherical harmonics involved are zonal, the work and the interpretation are much facilitated by the use of the current-function Ψ, whose value is given by (56).